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Skyrmion–anti-Skyrmion chains

Ya Shnir[1,2] **and D H Tchrakian**[1,2]

[1] Department of Computer Science, National University of Ireland Maynooth, Maynooth, Republic of Ireland
[2] School of Theoretical Physics—DIAS, 10 Burlington Road, Dublin 4, Republic of Ireland

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Abstract
Static axially symmetric sphaleron-type solutions describing chains of interpolating Skyrmion–anti-Skyrmions have been constructed numerically. The configurations are characterized by two integers n and m, where $\pm n$ are the winding numbers of the constituents Skyrmion and anti-Skyrmion, and the second integer m defines the type of solution—it has zero topological charge for even m and for odd values of m the Skyrmion–anti-Skyrmion chain has topological charge n. For the vanishing mass term we confirm the existence of such chain solutions for winding number $|n| \geqslant 2$. The similarity with monopole–anti-monopole pairs is highlighted.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The purpose of the present work is the construction of axially symmetric chain lump–anti-lump solutions in the usual Skyrme model [1] on $\mathbb{R}^3$. It was argued over the last decade that the multi-monopole (MM) solutions of the Yang–Mills–Higgs model and the multi-Skyrmion (MS) solutions of the Skyrme model have many features in common [2]. The question of constructing a Skyrme–anti-Skyrme pair (SAS) follows naturally from the corresponding construction of a monopole–anti-monopole (MAP) pair [3, 4] and monopole–anti-monopole chains [5], and like the former, it would describe a sphaleron-like configuration. Likewise unsurprisingly, the resulting solutions are not given in closed form but are constructed numerically. The construction of an axially symmetric Skyrme–anti-Skyrme pair was carried out recently by Krusch and Sutcliffe [7], who pointed out its possible physical relevance as a model for the deuteron. In that case, with only a particle–antiparticle pair, the axially symmetric solution would be expected to be of the minimal energy configuration, consistent with its interpretation as a model for the deuteron.

This task was carried out using a gradient flow technique in [7]. Given the considerable numerical complexity of this problem relative to the corresponding monopole [4, 5] one, it is in order to repeat it using a different numerical technique. This is done here, applying instead a boundary value procedure. We extended the analysis presented in [7] by inclusion of the pion-mass term in the Lagrangian of the model. Besides the Skyrme–anti-Skyrme pair solution, we constructed new axially symmetric saddle-point solutions, which represent chains of m single Skyrmions and anti-Skyrmions, each carrying charge n in alternating order. For an equal number of Skyrmions and anti-Skyrmions, the chains reside in the topologically trivial sector. When the number of Skyrmions exceeds the number of anti-Skyrmions by one, the chains reside in the sector with topological charge n. The aim here is to reveal qualitative and quantitative similarities of chains of Skyrmions and anti-Skyrmions, with those of chains of monopoles and anti-monopoles. Unlike for the $m = 2$ chain in [7] however, we cannot expect that the axially symmetric chain configurations with more numerous ($m \geqslant 2$) constituents we construct are of the minimal energy. Such sphaleron of the lowest energy may well exhibit discrete symmetries.

The boundary value problem we consider applies to a two-dimensional nonlinear partial differential equation system and the numerical technique employed is exactly that applied in the corresponding monopole problem [4, 5]. The two-dimensional system of equations is the result of the imposition of *axial symmetry* to this problem. We recover of course the MSs carrying topological charge n, which were found a long time ago for $n = 2$ in [9, 11] and for up to $n = 5$ in [10].

Before starting the analysis, it should be pointed out that certain qualitative features expected here are more similar to instanton solutions of the Yang–Mills system, rather than the corresponding features of the monopole solutions of the Yang–Mills–Higgs system, in spite of the fact that the latter are described on $\mathbb{R}^3$ like the Skyrmion, while the former are described on $\mathbb{R}^4$. There is a fundamental qualitative difference between 'instanton'- and 'monopole'-type solutions, namely that the gauge connection in the former is asymptotically *pure gauge* while in the latter case it is *one-half pure gauge*. It turns out that the asymptotic properties of the solitons of sigma models are 'instanton' like, irrespective of dimensions, in that the *composite connection* is asymptotically *pure gauge*. Thus, we expect from the outset that the Skyrme–anti-Skyrme lumps should be more akin to the instanton–anti-instanton [12], rather than the monopole–anti-monopole. The property we have in mind is that there exists a monopole–anti-monopole consisting of a charge n and a charge $-n$ pair with $n = 1$, and with winding number $n > 1$ [13]. It turns out that the corresponding instanton–anti-instanton exists only for values $n \geqslant 2$. This was found in [12] only as a result of a numerical construction, which is in agreement with the same nonexistence result for $n = 1$ in the analytic proof for the existence of such nonselfdual lumps given in [14, 15]. Thus, we would not expect here to find a Skyrme–anti-Skyrme lump consisting of a charge 1 and charge -1 pair. We do however find such a pair when the pion-mass potential is introduced. We will later elaborate on this feature of the solutions that we find in our conclusions. In section 2 we impose axial symmetry and calculate the resulting reduced two-dimensional energy density functional whose second-order equations will be integrated, as well as the topological (baryonic) charges of these configurations. Our numerical results are presented in section 3, and our conclusions in section 4.

2. The model and imposition of symmetry

In terms of the order parameter multiplet $\phi^a = (\phi^\alpha, \phi^3, \phi^4)$, $\alpha = 1, 2$, of the nonlinear $O(4)$ sigma-model field subject to $|\phi^a|^2 = 1$, the rescaled static Hamiltonian of the Skyrme [1] model is expressed as

$$\mathcal{H}_{\rm stat} = |\partial_i \phi^a|^2 + \tfrac{1}{4}|\partial_{[i}\phi^a \partial_{j]}\phi^b|^2 + \mu^2(1 - \phi^3) \tag{1}$$

$i = x, y, z$, with the notation $[ij]$ implying antisymmetrization. Here $\mu^2(1-\phi^3)$ is the pion-mass term.

The energy density (1) is bounded from below by the topological charge density

$$\varrho_0 = \frac{1}{24\pi^2}\varepsilon_{ijk}\varepsilon^{abcd}\partial_i\phi^a\partial_j\phi^b\partial_k\phi^c\phi^d$$
$$= \frac{1}{24\pi^2}\varepsilon_{ijk}\varepsilon^{\alpha\beta\gamma}(\partial_i\phi^\alpha\partial_j\phi^\beta\partial_k\phi^\gamma\phi^4 - 3\partial_i\phi^\alpha\partial_j\phi^\beta\partial_k\phi^4\phi^\gamma), \tag{2}$$

whose integral is the integer winding number n, namely the baryon number. It is well known that this lower bound cannot be saturated and hence we are concerned only with the second-order Euler–Lagrange equations.

Next, we state the axially symmetric ansatz parametrized by two functions $f = f(\rho, z)$ and $g = g(\rho, z)$, $\rho^2 = x^2 + y^2$, and in terms of the two-component unit vector n^α imposing axial symmetry on the $O(4)$ sigma-model field $\phi^a = (\phi^\alpha, \phi^3, \phi^4)$

$$n^\alpha = \begin{pmatrix}\cos n\varphi \\ \sin n\varphi\end{pmatrix} \tag{3}$$

with the integer n counting the winding of the azimuthal angle φ, being the baryon number.

This ansatz is expressed compactly as

$$\begin{aligned}\phi^\alpha &= \sin f \sin g n^\alpha \equiv a(\rho, z)n^\alpha \\ \phi^3 &= \sin f \cos g \equiv b(\rho, z) \\ \phi^4 &= \cos f \equiv c(\rho, z)\end{aligned} \tag{4}$$

with the functions $f = f(\rho, z)$ and $g = g(\rho, z)$ dependent on the radial coordinate $\rho = \sqrt{|x_\alpha|^2}$ of the $\mathbb{R}^2$ subspace, with the index $i = 1, 2$, and $z = x_3$. $\hat{x}^\alpha = x^\alpha/\rho$ is the unit radius vector in the $\mathbb{R}^2$ subspace and n^α is the unit vector with vortex number n. Ansatz (4) has the same general structure as the parametrization used in [7]. Furthermore, one readily verifies that the parametrization (4) is consistent, i.e. the complete set of the field equations, which follows from the variation of the original action of the Skyrme model, is compatible with two equations which follow from variation of the reduced action on ansatz (4). In discussing the asymptotics we employ polar coordinates replacing $r = \sqrt{\rho^2 + z^2}$, $\theta = \arctan\frac{z}{\rho}$. It is also convenient to use the trigonometric parametrization of the Skyrme model in terms of the functions f, g to represent the energy functional of the model and the topological charge (baryon number) density [16], although it is not appropriate from the point of view of numerical calculations because of the numerical errors which originate from the disagreement between the boundary conditions on the angular-type function $g(r, \theta)$ on the ρ-axis and the boundary points $r = 0, \infty$, respectively[3].

Indeed, the reduced two-dimensional energy density functional, resulting from the imposition of axial symmetry stated in ansatz (4), is given by [16],

$$E = \frac{1}{r^2}\left((r\partial_r f)^2 + (\partial_\theta f)^2 + \left[(r\partial_r g)^2 + (\partial_\theta g)^2 + \frac{n^2\sin^2 g}{\sin^2\theta}\right]\sin^2 f\right)$$
$$+ \frac{\sin^2 f}{r^4}\left((r\partial_r f\partial_\theta g - r\partial_r g\partial_\theta f)^2 + \frac{n^2\sin^2 g}{\sin^2\theta}[(r\partial_r f)^2 + (r\sin f\partial_r g)^2\right.$$
$$\left. + (\partial_\theta f)^2 + (\sin f\partial_\theta g)^2]\right) + \mu^2(1-\cos f) \tag{5}$$

[3] For the MS solution it results in the appearance of step-function-like dependence at these points: $g(0, \theta) = g(\infty, \theta) = \pi\Theta(\theta - \pi/2)$, Θ being the step function.

or, equivalently

$$E = \left[1 + \left(\frac{na}{\rho}\right)^2\right]\left[\left(a_\rho^2 + b_\rho^2 + c_\rho^2\right) + \left(a_z^2 + b_z^2 + c_z^2\right)\right] \\ + \left(\frac{na}{\rho}\right)^2 + ((a_{[\rho}b_{z]})^2 + (b_{[\rho}c_{z]})^2 + (c_{[\rho}a_{z]})^2) + \mu^2(1-c) \tag{6}$$

where $a_{[\rho}\, b_{z]} \equiv \partial_\rho a \partial_z b - \partial_\rho b \partial_z a = a_\rho b_z - b_\rho a_z$. The latter truncated functional represents some modification of the $O(3)$ sigma model on the half-plane.

We will be choosing our boundary conditions such that the resulting multi-Skyrmion or the Skyrmion–anti-Skyrmion chain solution has the appropriate baryon charge. In the case of the multi-Skyrmions this is the (topological) winding number n appearing in (3) and (4). In the case of Skyrmion–anti-Skyrmion chains with an odd number of lumps the baryon number is again n, and it vanishes for Skyrmion–anti-Skyrmion chains with an even number of lumps. The topological charge is labelled with two distinct integers, the winding number n and a second integer m which specifies the asymptotic value of the function $g(r,\theta)$ in (4) when $r \to \infty$, namely

$$\lim_{r\to\infty} g(r,\theta) = m\theta. \tag{7}$$

The formula for the baryon number that we use is

$$B = \int \varrho_0 \mathrm{d}^3x = -\frac{1}{2}n[\cos m\theta]_{\theta=0}^{\theta=\pi} = \frac{1}{2}n[1-(-1)^m], \tag{8}$$

the configurations being classified according to the values of two integer numbers, n and m. The consideration above indicates that the case $m = 1$ corresponds to the (multi-) Skyrmions of topological charge n, while $m = 2$ yields a configuration with zero net topological charge consisting of two lumps and a sphaleron-like axially symmetric static solution of the Skyrme model, consisting of a charge n Skyrmion and a charge $-n$ anti-Skyrmion. More general, for odd values of m the winding number n coincides with the topological charge of the Skyrme field $B = n$ whereas even values of m correspond to the deformations of the topologically trivial sector. In the following, we shall see that the value of the integer m defines the number of the constituents of the configuration which can be identified with individual charge n Skyrmions and charge $-n$ anti-Skyrmions placed along the axis of symmetry in alternating order.

3. Numerical results

The Euler–Lagrange equations arising from the variations of (6) have been integrated by imposing the boundary conditions, which obey finite mass-energy and finite energy density conditions as well as regularity and symmetry requirements. Also the sigma-model constraint is imposed.

The numerical calculations are performed employing the package FIDISOL/CADSOL, based on the Newton–Raphson iterative procedure [17]. We solve the system of three coupled nonlinear partial differential equations numerically, on a non-equidistant grid in x and θ, employing the compact radial coordinate $x = r/(1+r) \in [0:1]$. Typical grids used have sizes 75×60.

Note that the parametrization in terms of the angular-type functions $f(r,\theta)$, and$g(r,\theta)$ is plagued by the disagreement between the boundary conditions we have to impose on the function $g(r,\theta)$ on the ρ-axis and the boundary points $r = 0, \infty$, respectively. Therefore, it will be correct to impose boundary conditions on the fields a, b, c as [7]

$$a|_{r=\infty} = 0, \qquad b|_{r=\infty} = 0, \qquad c|_{r=\infty} = 1 \tag{9}$$

at infinity, and for the multi-Skyrmions and Skyrmion–anti-Skyrmion chains with odd number of constituents we require

$$a|_{r=0} = 0, \qquad b|_{r=0} = 0, \qquad c|_{r=0} = -1, \tag{10}$$

at the origin[4]. For the Skyrmion–anti-Skyrmion pair and the chains with even number of constituents the Neumann boundary conditions must be imposed there on the fields b, c, i.e.

$$a|_{r=0} = 0, \qquad \partial_r b|_{r=0} = 0, \qquad \partial_r c|_{r=0} = 0. \tag{11}$$

The boundary conditions along the z-axis for odd and even values of the integer number m appearing in (8) are

$$\begin{aligned} &a|_{\theta=0} = 0, \qquad \partial_\theta b|_{\theta=0} = 0, \qquad \partial_\theta c|_{\theta=0} = 0; \\ &a|_{\theta=\pi} = 0, \qquad \partial_\theta b|_{\theta=\pi} = 0, \qquad \partial_\theta c|_{\theta=\pi} = 0. \end{aligned} \tag{12}$$

To produce a configuration with the correct topology and boundary condition which can be used as an input into the system of Euler–Lagrange equations, we implemented the following algorithm. The trigonometric parametrization of the triplet a, b, c given by ansatz (4) is used with the linear dependence of the profile function $f(x,\theta) = \pi(1-x), x \in [0,1]$ on the compact radial variable x and the linear dependence of the second angular function $g(x,\theta) = m\theta, \theta \in [0,\pi]$ on the polar angle θ. Note that we do not impose additional symmetry restrictions on the triplet of fields a, b, c in the input configuration; however, the resulting numerical solutions reveal such a discrete symmetry with respect to the reflection $z \to -z$. In addition to the solution of the boundary problem performed with the package FIDISOL/CADSOL, the gradient flow equations are solved to check the numerical results although in the latter case convergence is somewhat slower.

As a first step we reproduced the well-known results for charge n multi-Skyrmions [9–11] and for the charge 2 Skyrmion and charge -2 anti-Skyrmion pair [7]. Evidently, this algorithm can be implemented to construct Skyrmion–anti-Skyrmion chains, the configurations similar to the monopole–anti-monopole chains constructed in [3–5, 13]. For example, the value of the integer number $m = 3$ corresponds to the charge $-n$ anti-Skyrmion located at the origin and two charge n Skyrmions located on the z-axis symmetrically with respect to the xy-plane; the system with $m = 4$, which resides in the topologically trivial sector, corresponds to the chain of two SAS pairs alternating on the symmetry axis, etc.

Indeed, for the values of the winding number $m \geqslant 3$ in the input configuration we produce the Skyrmion–anti-Skyrmion chain solutions with winding numbers $n \geqslant 2$. The relative error is estimated to be lower than 10^{-3}. Another check of the correctness of our results was performed by verifying that the virial relation [18], namely the identity that ensues from the Derrick scaling requirement[5] , is satisfied. This was done both for multi-Skyrmions (MSs) of different topological charges and for the Skyrmion–anti-Skyrmion chain configurations.

Although the initial configuration satisfies the boundary condition (7), we do not impose it as a boundary condition in our numerical calculations; nevertheless the solutions asymptotically tend to satisfy (7).

The numerical algorithm we implemented to solve the boundary problem allows us to evaluate the dipole moment of the solutions. Indeed, the leading term in the asymptotic expansion of the axially symmetric multi-Skyrmion solution is a dipole [8], i.e.

$$\phi^3 \sim \frac{d\cos\theta}{r^2} + O(r^{-3}),$$

[4] Note that the boundary conditions (10) correspond to the negative values of the baryon number.

[5] In this case, the identity is $T = G - 3V$, consisting of the positive definite integrals $T = \int d^3x |\partial_i \phi^a|^2$, $G = \frac{1}{4}\int d^3x |\partial_{[i}\phi^a \partial_{j]}\phi^b|^2$ and $V = \mu^2 \int d^3x (1-\phi^3)$.

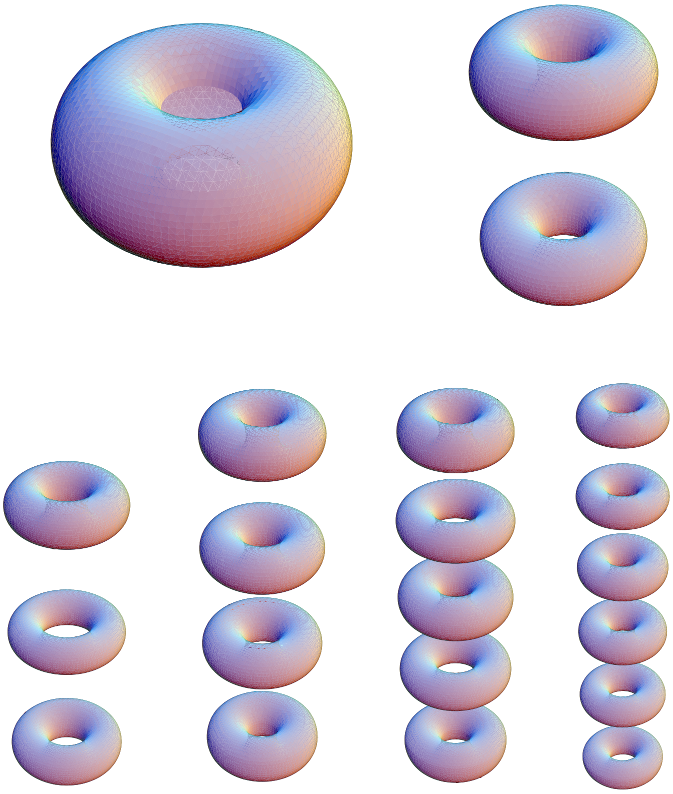

Figure 1. The 3D energy isosurfaces of the charge 2 axially symmetric Skyrmion and different Skyrmion–anti-Skyrmion chains for $n = 2, m = 1, \ldots, 6$ are shown for $\mu = 0$. (in different scales, cf table 1).

so in terms of the compact radial coordinate x we can extract the value of the dipole moment from the first and second derivatives of the field ϕ^3 at the boundary $x = 1$:

$$d = \partial_x \phi^3(\theta, x)\Big|_{x=1,\theta=0} + \frac{1}{2}\partial^2_{xx}\phi^3(\theta, x)\Big|_{x=1,\theta=0}.$$

Figure 1 displays the energy density surfaces for the charge 2 axially symmetric Skyrmion ($n = 2, m = 1$), charge 2 Skyrmion and charge -2 anti-Skyrmion pair ($n = 2, m = 2$) and for the Skyrmion–anti-Skyrmion chains where Skyrmions and anti-Skyrmions alternate along the symmetry axis ($n = 2, m = 3, 4, 5, 6$). Positions of the constituents can be identified as points in space where the field $\phi^4 \equiv c(\rho_0, z_0)$ is equal to -1 [7]. Indeed, figure 2 demonstrates that these points almost coincide with the maxima of the energy density distribution.

The higher energy Skyrmion–anti-Skyrmion chains are formed from m constituents of charge $\pm n$.

As an example, in table 1 we present the energy E of the Skyrmion–anti-Skyrmion chains with the winding number $n = 2$, the interaction energy which is defined as $\Delta E = mE^{(0)} - E$, where $E^{(0)} = 2.362$ is the energy of single charge $n = 2$ Skyrmion, the loci z_i of the field $c(0, z_i)$ and the numerical values of the dipole moment d for the solutions with θ winding number $1 \leqslant m \leqslant 6$.

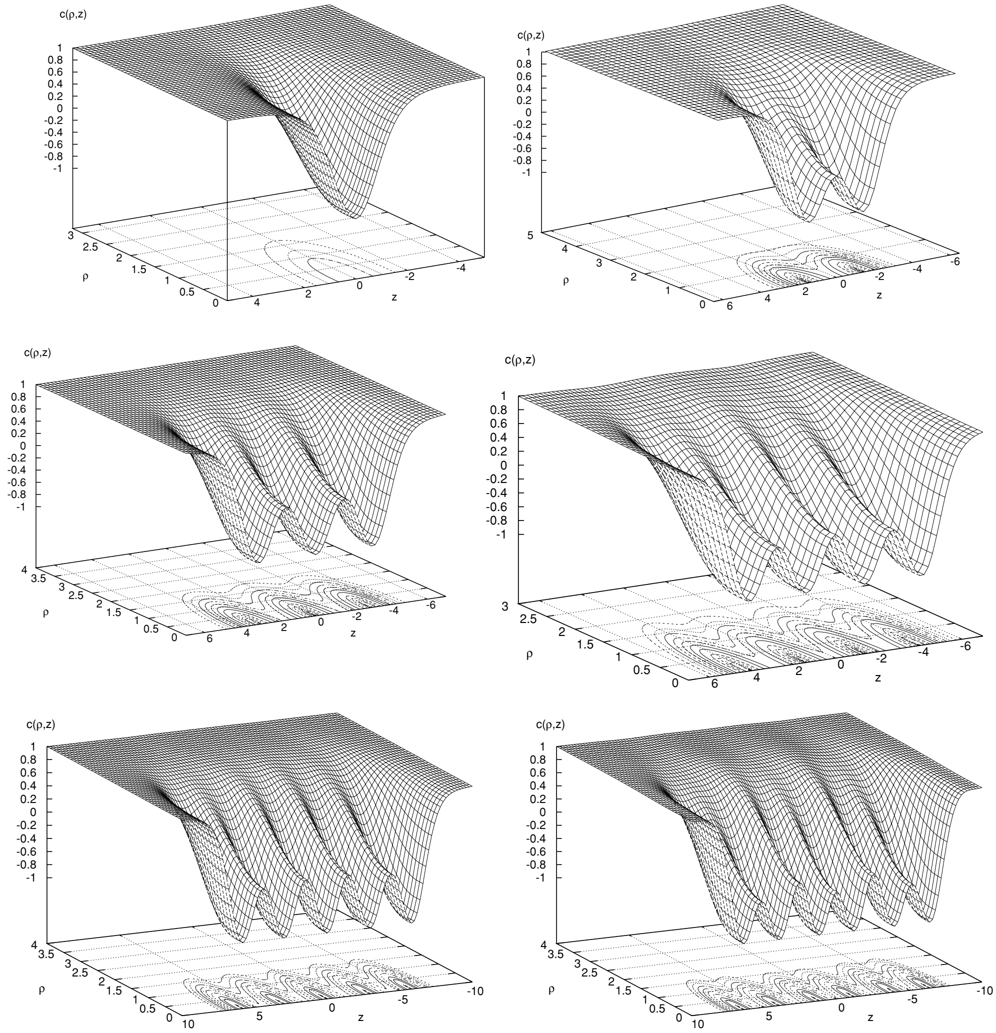

Figure 2. The field function $c(\rho, z)$ of the $m = 1$ charge 2 Skyrmion and Skyrmion–anti-Skyrmion chains for $n = 2, m = 1, \ldots, 6$ are shown for $\mu = 0$ as functions of the coordinates ρ, z.

A qualitative similarity between these Skyrmion–anti-Skyrmion chains and the monopole–anti-monopole chain solutions [5] is that in both cases there is a picture of an effective interaction between the constituents which allows the sphaleron-type solution to exist, although the nature of the interaction is different. In the former case it is an effective electromagnetic interaction between the constituents [5, 6] while in the latter case there is a dipole–dipole interaction between the Skyrmions [8]. We observe that the distances between positions of the constituents do not vary much within a chain. One can try to model the Skyrmion–anti-Skyrmion chains in the framework of the effective dipole–dipole interaction of the Skyrmions as was conjectured in [7].

Indeed the numerical results suggest that the dipole moment of the Skyrmion–anti-Skyrmion pair associated with the asymptotic behaviour of the field b is vanishing. We observe similar results for the chains with an even number of constituents. For odd values

Table 1. The energy of the Skyrmion–anti-Skyrmion chains, the positions of the constituents z_i and the values of the dipole moments are given for the $n = 2m$-chains with $m = 1, \ldots, 6$ for $\mu = 0$.

m	E	ΔE	z_i	d
1	2.36		0.0	4.31
2	4.64	0.08	± 1.44	0
3	6.94	0.14	0.0 ± 3.17	4.31
4	9.26	0.19	± 1.54 ± 4.45	0
5	11.57	0.24	0.0 ± 3.12 ± 6.05	4.28
6	13.88	0.29	± 1.62 ± 4.57 ± 7.42	0

Table 2. The values of the dipole moments of the $m = 1$ multi-Skyrmions with topological charge $n = 1, \ldots, 6$ and their masses are given for $\mu = 0$.

n	1	2	3	4	5	6
d	2.161	4.31	6.69	9.47	12.64	16.23
M	1.24	2.36	3.57	4.83	6.14	7.47

of m there is only one component of the dipole moment of the configuration, directed along the symmetry axis. For these chains it almost coincides with the dipole moment of the single charge n Skyrmion. Considering multi-Skyrmions of charge n, we can verify the conjecture of [7] concerning the additivity of the dipole moments. Indeed for the axially symmetric Skyrmions with topological charge $n = 1, \ldots, 4$ the dipole moment behaves approximately as $d = nd_1$ where $d_1 = 2.16$ is the dipole moment of the spherically symmetric charge one Skyrmion, although the results of the numerical calculations (cf table 2) indicate it is slightly higher than this additive estimate for $n \geqslant 4$. On the other hand it is known that the axially symmetric configurations of charge $n \geqslant 3$ are not the minimal energy states of the system [2].

Note that both types of the chain solutions with even and odd values of m possess the symmetry

$$(a(\rho, z), b(\rho, z), c(\rho, z)) \to (-a(\rho, z), -b(\rho, z), c(\rho, z))$$

which corresponds to the inversion of the asymptotic pion dipole fields.

For $n = 2, 3, 4$ these m-Skyrmion chains possess m points on the z-axis where the field $\phi^4(0, z_0) = c(0, z_0) = -1$ and a soliton is placed. Due to reflection symmetry, each such point $-z_0$ on the negative z-axis corresponds to a point z_0 on the positive z-axis. For even values of m the triplet of the fields a, b, c has reflection symmetry with respect to the xy-plane

$$(a(\rho, z), b(\rho, z), c(\rho, z)) \to (-a(\rho, -z), b(\rho, -z), c(\rho, -z)),$$

while for odd values of of m the reflection symmetry is

$$(a(\rho, z), b(\rho, z), c(\rho, z)) \to (a(\rho, -z), -b(\rho, -z), c(\rho, -z)).$$

Note that these symmetries are not imposed by the boundary conditions on the fields but arise when an initial configuration relaxes into a solution. For even m the field $c(0, 0)$ is far from the value -1 at the origin, although its value there decreases with increasing n.

As the winding number n increases further to $n \geqslant 5$, the positions of the minima of the field $c = -1$ are shifted away from the symmetry axis forming a system of concentric rings.

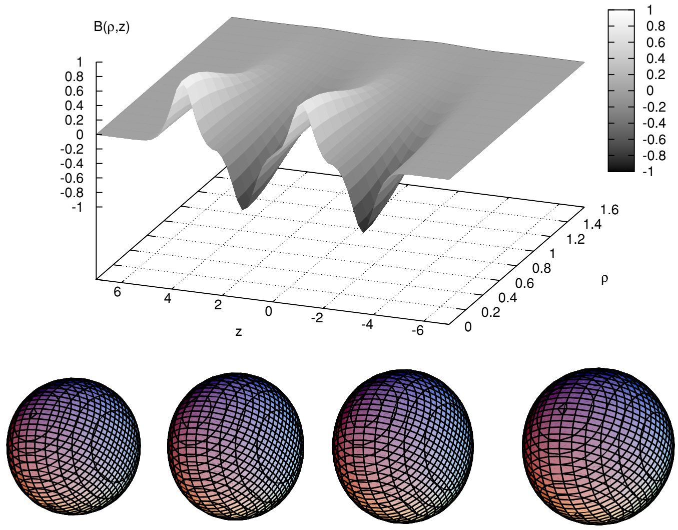

Figure 3. The topological charge density (top) $B(\rho, z)$ and the energy density isosurface of the $m = 4$ charge 1 Skyrmion and Skyrmion–anti-Skyrmion chain are shown for $\mu = 0.1$.

We observe that the radii of these rings increase with increasing n, the number of rings and their structure depending both on the winding number n and on m, for example, for a Skyrmion–anti-Skyrmion pair a single ring in the xy symmetry plane is formed [7]. For the configuration with $n = 6, m = 3$ we observe one ring $\left(\rho_0^{(1)} = 2.02, z_0^{(1)} = 0\right)$ on the symmetry plane, and two other rings placed symmetrically above and below of it, $\left(\rho_0^{(2,3)} = 1.83, z_0^{(2,3)} = \pm 1.41\right)$.

Inclusion of the pion-mass term in the Lagrangian (1) makes it possible for $n = 1$ Skyrmion–anti-Skyrmion chains to exist. In figure 3 we present the distribution of the topological charge density and the isosurface of the energy density for such a chain with $m = 4$ at $\mu = 0.1$. Note that such solutions do not exist either in Yang–Mills theory, nor in Skyrme theory in the absence of a pion-mass potential. In these cases only topological charge pairs of $n \geqslant 2$ and $n \leqslant -2$ are found.

An interesting analogy with the multi-monopoles (MM) and the monopole–anti-monopole (MA) chain solutions is that, the component of the Skyrme field ϕ^4 shows a clear relation to the corresponding behaviour of the magnitude $|\Phi|$ of the Higgs field of MMs and MAP pair, respectively [5], as seen in the plots in figure 2. Indeed, both these systems are parametrized by two integer numbers, one of those being associated with the topological charge of the constituents and corresponds to the n-fold rotation about the symmetry axis, and the other with the winding number, m, appears in the boundary conditions imposed on the field configuration. More specifically, in order to construct the MA chains we impose asymptotic boundary conditions on the YMH system which corresponds to the m-fold rotation of the fields as the polar angle θ varies from 0 to π [5]. Similarly, we impose the boundary condition $g(x, \pi) = m\pi$ on the angular function in ansatz (4) to construct the Skyrmion–anti-Skyrmion chains. Furthermore, both in the YMH model and in the Skyrme model the structure of the configurations changes as the winding number n increases beyond some critical value at which the positions of the constituents are no longer associated with some set of isolated points on the symmetry axis but form circles around it. Note that the similarity between the monopole–anti-monopole chains and chain solutions in the Skyrme model becomes even more transparent if we consider axially symmetric YM caloron solutions where monopoles and anti-monopoles are constituents of the caloron [19]. Then there is a holonomy operator $\mathrm{Tr}\mathcal{P}(\vec{r}) = \cos ||A_0(\vec{r})||$ whose loci $\mathcal{P}(\vec{r}_0) = -1$ are associated with the maxima of the action density [20].

4. Conclusions

We have constructed new static axially symmetric Skyrmion–anti-Skyrmion chain solutions numerically. This was done by solving the two-dimensional nonlinear partial differential equations as a boundary value problem, using the same formalism and numerical techniques as were used in the construction of the monopole–anti-monopole chains [4, 5] and non-self-dual instantons [12]. In implementing the requisite boundary conditions for this task, two integers (m, n) are employed. Both these examples are solved as two-dimensional problems, the former arising from the imposition of axial symmetry in $\mathbb{R}^3$ and the latter of bi-azimuthal symmetry in $\mathbb{R}^4$. The integer n labels the topological charge of each constituent lump or anti-lump. In the monopole case [4, 5] n is the winding of the azimuthal angle in $\mathbb{R}^3$, while in the instanton case [12] it is the winding of the two azimuthal angles in $\mathbb{R}^4$, taken to be equal. In both cases, it is the topological charge descending from the second Chern–Pontryagin charge subject to the respective symmetries. The integer m enters the asymptotic value of the (form factor) function which maps on to the remaining angular coordinate, namely the first (polar) angle different from the azimuthal angle(s). m is not a topological charge. The total topological charge of any such finite action/energy configuration vanishes when m is *even*, and equals n when m is *odd*.

In the present work, we have restricted to $m = 2, \ldots, 6$. Clearly, for $m = 1$ we simply recover the axially symmetric MS of charge n, which we have done as a warmup. We have examined the cases with values of n, starting from $n = 1$, through to $n = 6$.

Our numerical investigations indicate fairly clearly that in the usual Skyrme model with no pion-mass term, there exist no zero baryon number solutions when each of the constituents carries baryon number $|n| < 2$. However, inclusion of the pion-mass term results in the existence of chains with $n = 1$. This is perhaps not surprising in the background of the known close analogy between solutions to Yang–Mills and sigma models respectively, e.g. the usual YM model and the Skyrme model with no pion-mass term. This analogy is demonstrated most simply by considering the unit charge solutions (BPST instantons) of the YM system parametrized by the radial function $w(r)$ on the one hand, and the unit charge solutions (hedgehog) of the Skyrme system parametrized by the radial chiral function $f(r)$ on the other. This correspondence is $w = \cos f$, and it manifests itself already in the one-dimensional reduced actions of the two systems, but only in the absence of the pion-mass potential in the (Skyrme) sigma model. This is clear since the one-dimensional reduced action does not feature a pion-mass-like term $(1 - \cos f) \equiv (1 - w)$, in the presence of which the analogy between YM instantons and Skyrmions disappears.

The $n = 2$, $\mu = 0$ example was the one examined most intensively, since it is the first non-marginal case where we could reliably verify the existence of the Skyrmion-anti-Skyrmion chains. However, unlike in the two previous examples in [4, 5] and in [12], the binding energy of the constituents of the Skyrmion chain turned out to be quite weak, so the $n = 2$ chains, especially for the odd numbers of the consituents, are very unstable w.r.t. perturbations.

The nonexistence of the Skyrmion–anti-Skyrmion chains with constituents carrying baryon numbers $\pm n$, for values of $n \leqslant n_{\mathrm{min}}$ is not surprising. In the corresponding analytic proof [14, 15] of existence for non–self-dual instantons of zero Pontyagin charge, the existence of the case where the constituents carried Pontyagin charge $n = |1|$ was not established, indicating its nonexistence. In that case $|n_{\min}| = 2$ which coincides with previous result of [7].

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